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EFFICIENT ALGORITHMS FOR PARALLELIZING TRIDIAGONAL SYSTEMS OF EQUATIONS

Abstract. The article is devoted to the development of the maximal parallel forms of mathematical models with a tridiagonal structure. The example of solving the Dirichlet and Neumann problems by the method of straight lines and the sweep method for the heat equation illustrates the direct fundamental features of constructing parallel algorithms. It is noted that the study of the heat and mass transfer processes is run through their numerical modeling based on modern computer technology.

It is shown that with the multiprocessor computing systems' development, there disappear the problems of increasing their peak performance. On the other hand, building such systems, as a rule, requires standard network technologies, mass-produced processors, and free software. The noted circumstances aim at solving the so-called big problems.

It should be borne in mind that the classical approach to solving the tridiagonal structure models based on multiprocessor computing systems is far more time-consuming compared to single-processor computing facilities. That is explained by the recurrence relations that make the basis of classical methods. Therefore, the proposed studies are relevant and aim at the distributed algorithms development for solving applied problems.

The proposed research aims to construct the maximal parallel forms of mathematical models with a tridiagonal structure.

The paper proposes the schemes to implement parallelization algorithms for applied problems and their mapping to parallel computing systems.

Parallelization of tridiagonal mathematical models by the method of straight lines and the sweeping method allows designing absolutely stable algorithms with the maximum parallel form and, therefore, the minimum possible time for their implementation on parallel computing devices. It is noteworthy that in the proposed algorithms, the computational errors of the input data are separated from the round-off errors inherent in a PC.

The proposed approach can be used in various branches of metallurgical, thermal physics, economics, and ecology problems in the metallurgical industry.

Keywords: multiprocessor system, tridiagonal structure, method of straight lines, algorithms, thermal physics, parallel forms.

Raising of problem and analysis of the last achievements. The parallel computation systems are quickly developing and when computer clusters came into

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service, parallel computations became available for many people. To construct clusters the mass processors, standard net technologies and freely distributed software support are used as a rule. Let's note that in the guise of computer cluster they implicate a computer body being united in the framework of a certain net to solve one problem. A small computer cluster consisting of 6 personal computers may be efficiently used by small departments. And at present, in this connection, the problem of efficient software development turned to be one of the central problems of parallel computations in the whole. Creation of parallel computation systems necessitated the designing of mathematical conceptions of parallel algorithm development, that is, the algorithms adapted to realization on similar computation systems [1 – 6]. On the other hand the development of mathematical modeling and practice needs gives rise to more complicated models. To understand them and to develop the analysis principles one has to utilize the newest achievements from quite different mathematical fields. Problem discretization results in equation systems with a great number of unknowns. Former methods of their solution are not always suitable from the point of view of accuracy, rate, required memory, algorithm structure, and such like considerations. The new ideas in the field of computational mathematics arise and are realized. In the final analysis, the new methods of numerical experiments realization are as well created for more perfect mathematical models [7,8]. In this work the efficacy of system of linear algebraic equations (SLAE) parallelization of three-diagonal structure using the numerical-and-analytical method of lines is shown on the pattern of the simplest heat conductivity problem solution.

The purpose of the research is to construct the most parallel forms of mathematical models of a tridiagonal structure. The application of the numerical and analytical direct method and sweep methods for such systems parallelizing allow designing its solutions having the maximum parallel form and, therefore, the least time for its implementation on parallel computing devices.

Mathematical problem setting. Let's examine a boundary-value Dirichlet's problem solution for a one-dimensional heat conduction equation.

$$\frac{\partial \mathrm{Y}}{\partial t} = a\frac{\partial^2 \mathrm{Y}}{\partial x^2}, \quad t \in [t_0, T], \, x \in [x_0, x_L] \tag{1}$$

with the initial condition

$$\mathrm{Y}_{|t=t_0} = \varphi(x) \tag{2}$$

and the boundary data

$$\mathrm{Y}_{|\, x=x_0} = \mathrm{Y}W(t), \; Y_{|x=x_L} = \mathrm{Y}L\ (t), \tag{3}$$

Let's juxtapose a net domain to a definitional one of sought function Y(t, x) in (1) – (3) problem

$$\left.\begin{array}{l} t_y = \Upsilon \times Dt1,\ \Upsilon = \overline{\lambda M},\ Dt1 = T / M,\ M \in Z \\ x_p = p \times Dt1,\ p = \overline{0,2m},\ Dt1 = (x_L - x_0) / 2m,\ m \in Z \end{array}\right\}, \quad (4)$$

Let's introduce the new unit-normalized independent arguments

$$\left.\begin{array}{l} \varepsilon_t = \dfrac{t - t_{j-1}}{t_j - t_{j-1}} \in [0,1] \\ \varepsilon_x = \dfrac{x - x_p}{x_{p+1} - x_p} \in [-1,1] \end{array}\right\}, \quad (5)$$

here we'll get:

$$\frac{\partial \Upsilon_{p+\varepsilon_{xy},1}}{\partial \varepsilon_t} = A \frac{\partial^2 \Upsilon_{p+\varepsilon_{xy},1}}{\partial \varepsilon_x^2}, \qquad A = Dt1 \frac{a}{Dx1^2}, \quad (6)$$

where $\Upsilon_{p+\varepsilon_x,1}(\varepsilon_t, \varepsilon_x)$ is sought piecewise-analytic function.

The research results

Discretization scheme by method of lines. The conception of (1) – (3) problem discretization by method of lines lies in the following [8]. After the equation (6) finite-difference approximation on a temporary variable we'll get
the second-order system of ordinary differential equations (SODE).

$$\Upsilon^{//}_{p+\varepsilon_x,1}(\varepsilon_x) - \frac{1}{A}\Upsilon_{p+\varepsilon_x,1}(\varepsilon_x) = -\frac{1}{A}\Upsilon O_{p+\varepsilon_x,1}(\varepsilon_x), \quad (7)$$

where $YO_{p+\varepsilon x,1}(\varepsilon_x)$ is known initial function.

General solution of equation (7) seems to be a ratio of the following type:

$$Y_{p+\varepsilon x,1}(\varepsilon_x) = Y^*_{p+\varepsilon_x,1}(\varepsilon_x) + C_p c\eta\beta(\varepsilon_x) + D_p S\eta\beta(\varepsilon_x), \quad (8)$$

where C_p, D_p are the constants of integration;
$YO_{p+\varepsilon x,1}(\varepsilon_x)$ - specific solution of heterogeneous equation (7);

$\beta = \sqrt{\dfrac{1}{A}}$ – proper numbers of characteristic equation

$$\beta^2 - \frac{1}{A} = \Delta. \quad (9)$$

While differentiating solution (8) on ε_x, they get

$$\beta^2 - \frac{1}{A} = \Delta, \quad (9)$$

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where $Y_{p+\varepsilon x,2}(\varepsilon_x) = Dx1\frac{Y_{p+\varepsilon_{x,1}}}{x}\Big|_{t=t_j}$ is an unknown gradient function.

Determining constants of integration in ratios (8), (10) from the conditions when $\varepsilon_x = \pm 1$:

$$\mathrm{Y}_{P+\varepsilon x,1}(E_x)\big|_{\varepsilon_x=\pm1} = \mathrm{Y}_{p\pm1} \tag{11}$$

they get

$$\mathrm{Y}_{з+\varepsilon x,1}(\varepsilon_x) = \left\{ \begin{array}{l} \mathrm{Y}^*_{з+\varepsilon_{x,1}}(\varepsilon_x) + \frac{S\eta\beta(1+\varepsilon_x)}{S\eta\beta(2)}\left[\mathrm{Y}_{p+1,1} - \mathrm{Y}^*_{p+1,1}\right] + \\ + \frac{S\eta\beta(1-\varepsilon_x)}{S\eta\beta(2)}\left[\mathrm{Y}_{p-1,1} - \mathrm{Y}^*_{p-1,1}\right] \end{array} \right\}, \tag{12}$$

$$\mathrm{Y}_{з+\varepsilon x,2}(\varepsilon_x) = \left\{ \begin{array}{l} \mathrm{Y}^*_{з+\varepsilon_{x,2}}(\varepsilon_x) + \frac{S\eta\beta(1+\varepsilon_x)}{S\eta\beta(2)}\left[\mathrm{Y}_{p+1,2} - \mathrm{Y}^*_{p+1,2}\right] + \\ + \frac{S\eta\beta(1-\varepsilon_x)}{S\eta\beta(2)}\left[\mathrm{Y}_{p-1,2} - \mathrm{Y}^*_{p-1,2}\right] \end{array} \right\}, \tag{13}$$

having put $\varepsilon_x = 0$, let's pass from distributed solution form (12), (13) to their discrete analogues in the form of SLAE:

$$C_p \begin{Bmatrix} \mathrm{Y}_{p+1,1} \\ \mathrm{Y}_{p+1,2} \end{Bmatrix} - \begin{Bmatrix} \mathrm{Y}_{p,1} \\ \mathrm{Y}_{p,2} \end{Bmatrix} + D_p \begin{Bmatrix} \mathrm{Y}_{p-1,1} \\ \mathrm{Y}_{p-1,2} \end{Bmatrix} = \begin{Bmatrix} f_{p,1} \\ f_{p,2} \end{Bmatrix}, \tag{14}$$

where

$$\left. \begin{array}{l} C_p = \frac{S\eta\beta(1)}{S\eta\beta(2)}, \qquad D_p = \frac{Su\beta(1)}{Su\beta(2)} \\ \begin{Bmatrix} f_{p,1} \\ f_{p,2} \end{Bmatrix} = C_p \begin{Bmatrix} \mathrm{Y}^*_{p+1,1} \\ \mathrm{Y}^*_{p+1,2} \end{Bmatrix} - Y^*_{p,\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}} + D_p \begin{Bmatrix} \mathrm{Y}^*_{p,1} \\ \mathrm{Y}^*_{p,2} \end{Bmatrix} \end{array} \right\}, \quad p = \overline{1,2m-1} \quad , \tag{15}$$

SLAE (14) are invariant relative to net functions $\mathrm{Y}_{p,1}$ and $\mathrm{Y}_{p,2}$ and have three-diagonal structure where boundary elements $\mathrm{Y}_{0.1}$, $\mathrm{Y}_{2m,1}$ for problem (1) – (3) are identical to the values of boundary functions (3):

$$\mathrm{Y}_{0,1} = YW(t_j), \quad \mathrm{Y}_{2m,1} = WL(t_j). \tag{16}$$

Identically, $Y_{0.2}$, $Y_{2m,2}$ elements on the domain boundary in SLAE (14) take the following values:

$$\left. \begin{array}{l} \mathrm{Y}_{0,2} = Dx1 \times \frac{\partial \mathrm{Y}}{\partial x}\Big|_{\substack{x=x_0 \\ t=t_{oj}}} \\ \mathrm{Y}_{2m,2} = Dx1 \times \frac{\partial \mathrm{Y}}{\partial x}\Big|_{\substack{x=x_L \\ t=t_j}} \end{array} \right\}, \tag{17}$$

which in Neumann's problems correspond to second-order boundary data.

In so doing, the mathematical model invariance in the form of SLAE (14) relative to $Y_{p,1}$, $Y_{p,2}$, net functions seemingly, reflects deeper group-theoretical features of input equation (1) relative to Coshi's data as well. In light of noted, the differential manifold and Dirichlet's and Neumann's problems group classification being carried out on Coshi's data in SLAE permits to continue the analysis on the pattern of only one group of decision variables $Y_{p,1}$ examination.

Sweep method analysis. Let's describe a simple and ordinary solving method of SLAE (14) having got the name of

the sweep method [1]. Let's postulate the existence of such two vectors *E* and *G* that for wha

$Y_{p,1}$ (p = 1,2m -1) the following equality is executed:

$$Y_{p,1} = E_p Y_{p+1,1} + G_p . \tag{18}$$

Having reduced into (18) index p by a unit, they get

$$Y_{p-1,1} = E_{p-1} Y_{p,1} + G_{p-1} . \tag{19}$$

After ratio (19) substitution to SLAE (14) we'll find

$$Y_{p,1} = \frac{C_p}{1 - D_p E_{p-1}} Y_{p+1,1} + \frac{D_p G_{p-1} - f_{p,1}}{1 - D_p E_{p-1}} . \tag{20}$$

Comparing equations (18) and (20) and noting that both of the equations being equitable for all the $p = \overline{1,2m-1}$ indices they get the recurrent ratios:

$$E_p = \frac{C_p}{1 - D_p E_{p-1}}, \quad G_p = \frac{D_p G_{p-1} - f_{p,1}}{1 - D_p E_{p-1}}, \tag{21}$$

realizing the algorithm of direct sweep. In fact, the following goes from the conditions on the left boundary (3):

$$E_0 = 0, \quad G_0 = Y_{0,1} = YW(t_j) . \tag{22}$$

Further E_p, G_p elements are calculated in all the points in increment direction.

$Y_{p,1}$ (p = 1.2m -1) on the recurrent formulae (21). Then it goes from the right boundary condition (3) that $Y_{2m,1} = YL(t_j)$. This provides for start of reverse sweep on the recurrent formula (19) in p decrease direction from p=2m-1 up p=1.

If a sweep algorithm on formulae (20), (18) has the right orientation, it is obvious that it is possible to organize a sweep algorithm of the opposite left orientation as well. Let's assume the existence of such *E* and *G* vectors that for all the $Y_{p,1}$ the following equality is executed:

$$Y_{p,1} = E_p Y_{p-1,1} G_p . \tag{22}$$

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Having changed in (22) index p by increasing per a unit, they get

$$\mathrm{Y}_{p+1,1} = E_{p+1}\mathrm{Y}_{p,1}G_{p+1}. \qquad (23)$$

Then after elimination from SLAE (14) $Y_{p+1,1}$ variables by substitution on formulae (23), it seems possible to develop the following recurrent dependencies:

$$E_p = \frac{D_p}{1 - C_p E_{p+1}}, \quad G_p = \frac{C_p G_{p+1} - f_{p,1}}{1 - C_p E_{p-1}} \qquad (24)$$

corresponding to the right sweep algorithm in index p from p=2m-1 up p=1 decrease direction. Here the reverse sweep is realized on the recurrent ratios (22) in increment direction of $p = \overline{1,2m-1}$ index, which was to be proved.

Let's note that to calculate using sweep method system solution (14), consisting of (2m+1) equations, it's necessary to carry through arithmetical operations in the quantity being only by finite number times bigger than unknown quantity. To solve equations with *N* unknown quantities of arbitrary linear system N using method of exclusion one usually has to use up arithmetic in N^3 quantity. They managed to attain such a reduction of arithmetic number when system (14) solving by sweep method while successfully using the specific character of this system.

Conclusion. Three-diagonal SLAE parallelization being realized on the ground of numerical-and-analytical method of lines permits synthesizing of absolutely stable algorithms having maximal parallel form and, thereby, minimal possible time for its realization on parallel computing devices. Noteworthy is also the fact that input data computing errors being in it separated from round-off ones thus inherent in real personal electronic computers. The algorithm is illustrated on the pattern of initial-boundary-value Dirichlet's problem solution for heat conduction equation. But in this connection, it turns out to be an invariant one for coset problems – Neumann's ones too. This fact obviously reflects deeper group-theoretic properties of parabolic equations as well relative to Coshi's data on spatial variable.

The proposed approach can be used in various branches of metallurgical, thermal physics, economics, and ecology problems in the metallurgical industry.

Keywords: multiprocessor system, tridiagonal structure, method of straight lines, algorithms, thermal physics, parallel forms.

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Ефективні алгоритми розпараалелювання тридіагональних систем рівнянь

Статтю присвячено розробці максимально паралельних форм математичних моделей, які мають тридіагональну структуру. Безпосередньо принципові особливості конструювання паралельних алгоритмів ілюструються на прикладі розв'язків задач Діріхле та Неймана методом прямих і методом прогонки для рівнянь теплопровідності. Відзначається, що дослідження процесів тепло - і масообміну реалізується шляхом їх числового моделювання на основі застосування сучасних засобів обчислювальної техніки.

Показано, що з розвитком багатопроцесорних обчислювальних систем зникають проблеми в збільшенні їх пікової продуктивності. З іншого боку, для побудови таких систем, як правило, застосовують стандартні мережеві технології, процесори масового

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виробництва, вільне програмне забезпечення. Зазначені обставини й спрямовані на розв'язок так званих великих задач.

При цьому необхідно мати на увазі, що застосування класичного підходу до розв'язку моделей тридіагональних структур на основі використання багатопроцесорних обчислювальних систем відзначаються набагато більшими часовими витратами в порівнянні з однопроцесорними обчислювальними засобами. Це пояснюється застосуванням рекурентних співвідношень, які покладені в основу класичних методів. Отже, для паралельних обчислювальних систем необхідно окремо конструювати процес векторизації обчислень. У зв'язку з цим запропоновані дослідження є актуальними і спрямованими на розвиток розподілених алгоритмів розв'язку прикладних задач.

Мета запропонованих досліджень полягає в конструюванні максимально паралельних форм математичних моделей тридіагональних структур. Запропоновані в даній роботі схеми спрямовані на реалізацію алгоритмів розпаралелювання прикладних задач і їх відображення на паралельні обчислювальні системи.

Розпаралелювання тридіагональних математичних моделей методом прямих і методом прогонки дозволяє конструювати абсолютно стійкі алгоритми, що мають максимальну паралельну форму і, отже, мінімально можливий час їх реалізації на паралельних обчислювальних пристроях. Помітимо, що в запропонованих алгоритмах відокремлено обчислювальні похибки вхідних даних від похибок округлення, властиві ПЕОМ.

Запропонований підхід може бути використаним в різних галузях металургійної теплофізики, економіки, а також задач екології металургійної промисловості.

Efficient algorithms for parallelizing tridiagonal systems of equations

The article is devoted to the development of the maximal parallel forms of mathematical models with a tridiagonal structure. The example of solving the Dirichlet and Neumann problems by the method of straight lines and the sweep method for the heat equation illustrates the direct fundamental features of constructing parallel algorithms. It is noted that the study of the heat and mass transfer processes is run through their numerical modeling based on modern computer technology.

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